Bankruptcy Problem under Uncertainty of Claims and Estate

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IWoMCDM – 2021

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Motivation - Introduction

- Several individuals hold claims on a finite resource - estate and the total amount is not enough to fulfill all of the claims
- **Problem: Distribute the resource (Estate) to individual claimants fairly so as to respect individual claims as much as possible**

Classical bankruptcy problems (CB) and games

CBP - triple (N, c, E):

$N = \{1,2,\ldots,n\}$ – set of claimants

$\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ – positive vector of claims c_i , $i \in N$

E - positive total estate.

Alternatively:

($\boldsymbol{c}$; E) generates a cooperative game (N; v), - *bankruptcy game*, whose characteristic form is given by

$v(T) = max\{0, E - \Sigma c_i\}$, $T \subseteq N$ - *value* of coalition T

Interval bankruptcy problem 1

- $I(\mathbf{R})$ - set of all closed and bounded intervals on **R**
- **R -** set of real numbers
- $I(\mathbf{R})^n$ - set of all n-dimensional vectors in $I(\mathbf{R})$
- $I, J \in I(\mathbf{R})$, with $I = [I^-; I^+]$, $J = [J^-; J^+]$ and $k \geq 0$
- Interval operations:

  $I + J = [I^- + J^-; I^+ + J^+]$, $kI = [kI^-; kI^+]$
- Partial ordering on $I(\mathbf{R})^n$:

  $I \leq J$ if $I^- \leq J^-$ and $I^+ \leq J^+$

  $I = J$, if $I \leq J$ and $J \leq I$, $I < J$, if $I \leq J$ and $I \neq J$

For any $T \subseteq N$, we use the notation:

$$c^-(T) = \sum_{i \in T} c_i^- , \quad c^+(T) = \sum_{i \in T} c_i^+$$

Minimal/maximal rights:

$$m_i^-(e) = max\{c_i^-, e - c^+(N\setminus\{i\})\}, \quad m_i^+(e) = min\{c_i^+, e - c^-(N\setminus\{i\})\}$$

Interval bankruptcy problem 2

Definition 1: A *bankruptcy rule* for an IB-problem (c,E) is a vector mapping $\mathbf{s} : I(\mathbf{R}^{+})^{n+1} \rightarrow I(\mathbf{R}^{+})^{n}$
where $\mathbf{s}(c, E) = (s_1(c, E), \ldots, s_n(c, E))$,
such that $c = (c_1,\ldots,c_n) \in I(\mathbf{R}^{+})^{n}$, $c_i = [c_i^{-}; c_i^{+}]$, $i \in N$,
and $E = [E^{-}; E^{+}] \in I(\mathbf{R}^{+})$, satisfying

(1) $s_i(\mathbf{c}, E) = [s_i^{-}(c,E), s_i^{+}(c,E)] \subseteq c_i = [c_i^{-}, c_i^{+}]$,
for all $i \in N$, (*Individual rationality*)

(2) $E = [E^{-}; E^{+}] \subseteq \sum_{j \in N} s_j(\mathbf{c}, E)$. (*Efficiency*)

Interval bankruptcy problem 3

Proposition 1: (c,E) - IB-problem. Let

$$c^-(N) \le E^- \le E^+ \le c^+(N).$$

Then $s_i(c,E) = [s_i^-; s_i^+] \in I(\mathbf{R}^+)$ defined for $i \in N$, by

$$s_i^- = m_i^-(E^-) + [m_i^+(E^-) - m_i^-(E^-)] \frac{E^- - m_N^-(E^-)}{m_N^+(E^-) - m_N^-(E^-)}, \quad (*)$$

$$s_i^+ = m_i^-(E^+) + [m_i^+(E^+) - m_i^-(E^+)] \frac{E^+ - m_N^-(E^+)}{m_N^+(E^+) - m_N^-(E^+)}, \quad (**)$$

is a bankruptcy rule called the *adjusted proportional rule* (*AP-rule*) *for the IB-problem* (**c**,E) satisfying conditions (1), (2) .

Fuzzy interval bankruptcy problem 1

- Claimants declare their claims with vague words: “about”, “around”, “rather small”, “very big”, etc.
- **The key issue**: how to distribute the uncertain, i.e. interval, fuzzy interval or, eventually, the estate given with some probability, to the individual claimants?
- A *fuzzy set A* of **R** is a *fuzzy number (fuzzy interval)*, whenever *A* is normal (i.e. there exists x_0 with $\mu_A(x_0) = 1$) and its membership function $\mu_A : \mathbf{R} \to [0;1]$ satisfies that the α-*cut* $[A]_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$ is closed, compact and convex subset of **R** for every $\alpha \in [0;1]$.
- Fuzzy number *A* of **R** is equivalent to the family of α-cuts $\{[A]_\alpha \mid \alpha \in [0;1]\}$.

Fuzzy interval bankruptcy problem 2

Definition 2: $\tilde{c} = (\tilde{c}_1,\ldots,\tilde{c}_n) \in F(\mathbf{R}^+)^n$ be a vector of fuzzy numbers:

$\tilde{c}_i = [c_i^-(\alpha); c_i^+(\alpha)]$, $i \in N$, $\widetilde{E} = [E^-(\alpha); E^+(\alpha)] \in F(\mathbf{R}^+)$

$\alpha \in [0;1]$ be the families of α-cuts

A *bankruptcy rule* for an FB-problem $(\tilde{\boldsymbol{c}},\widetilde{E})$ is a vector mapping $\tilde{\boldsymbol{s}} : F(\mathbf{R}^+)^{n+1} \to F(\mathbf{R}^+)^n$:

$[\tilde{\boldsymbol{s}}(\tilde{c},\widetilde{E})]_\alpha = ([\tilde{s}_1(\tilde{c},\widetilde{E})]_\alpha, \ldots, [\tilde{s}_n(\tilde{c},\widetilde{E})]_\alpha)$

where $\tilde{s}_i : F(\mathbf{R}^+)^{n+1} \to F(\mathbf{R}^+)$, $i \in N$.

Here, for each $\alpha \in [0;1]$, $[\tilde{\boldsymbol{s}}(\tilde{c},\widetilde{E})]_\alpha$ is an IB-problem.

Fuzzy interval bankruptcy problem 3

Proposition 2: Let $(\tilde{\boldsymbol{c}}; \widetilde{E})$ be a FB-problem. Let $\widetilde{E} = \{[E^-(\alpha); E^+(\alpha)] \mid \alpha \in [0;1]\}$ and let

$$\sum_{i \in S} c_i^-(\alpha) \le E^-(\alpha) \le E^+(\alpha) \le \sum_{i \in S} c_i^+(\alpha)$$

for all $\alpha \in [0;1]$.

Then for $\alpha \in [0;1]$, $[\tilde{s}_i(\tilde{\boldsymbol{c}}; \widetilde{E})]_\alpha = [s_i^-(\alpha); s_i^+(\alpha)] \in I(\mathbf{R}^+)$ is a closed interval defined for $i \in N$, by

$$s_i^-(\alpha) = m_i^-\left(E^-(\alpha)\right) + [m_i^+\left(E^-(\alpha)\right) - m_i^-\left(E^-(\alpha)\right)] \ \frac{E^-(\alpha) - m_N^-(E^-(\alpha))}{m_N^+\left(E^-(\alpha)\right) - m_N^-(E^-(\alpha))}, \qquad (+)$$

$$s_i^+(\alpha) = m_i^-\left(E^+(\alpha)\right) + [m_i^+\left(E^+(\alpha)\right) - m_i^-\left(E^+(\alpha)\right)] \ \frac{E^+(\alpha) - m_N^-(E^+(\alpha))}{m_N^+\left(E^+(\alpha)\right) - m_N^-(E^+(\alpha))}. \qquad (++)$$

Family $\{[s_i^-(\alpha); s_i^+(\alpha)] \mid \alpha \in [0;1]\}$, where the α-cuts are defined by (+), (++), defines a bankruptcy rule called the *adjusted fuzzy proportional rule* (*AFP-rule*) *for the FB-problem* $(\tilde{\boldsymbol{c}}; \widetilde{E})$.

Fuzzy interval bankruptcy problem 4

The mean values $s_i^-(\widetilde{E})$, $s_i^+(\widetilde{E})$ give the corresponding *interval* share $[s_i^-(\widetilde{E}); s_i^+(\widetilde{E})]$ of claimant *i*:

$$s_i^-(\widetilde{E}) = \frac{\int_0^1 \alpha s_i^-(\alpha)d\alpha}{\int_0^1 s_i^-(\alpha)d\alpha}\ , \quad s_i^+(\widetilde{E}) = \frac{\int_0^1 \alpha s_i^+(\alpha)d\alpha}{\int_0^1 s_i^+(\alpha)d\alpha},\ i \in N. \quad (2)$$

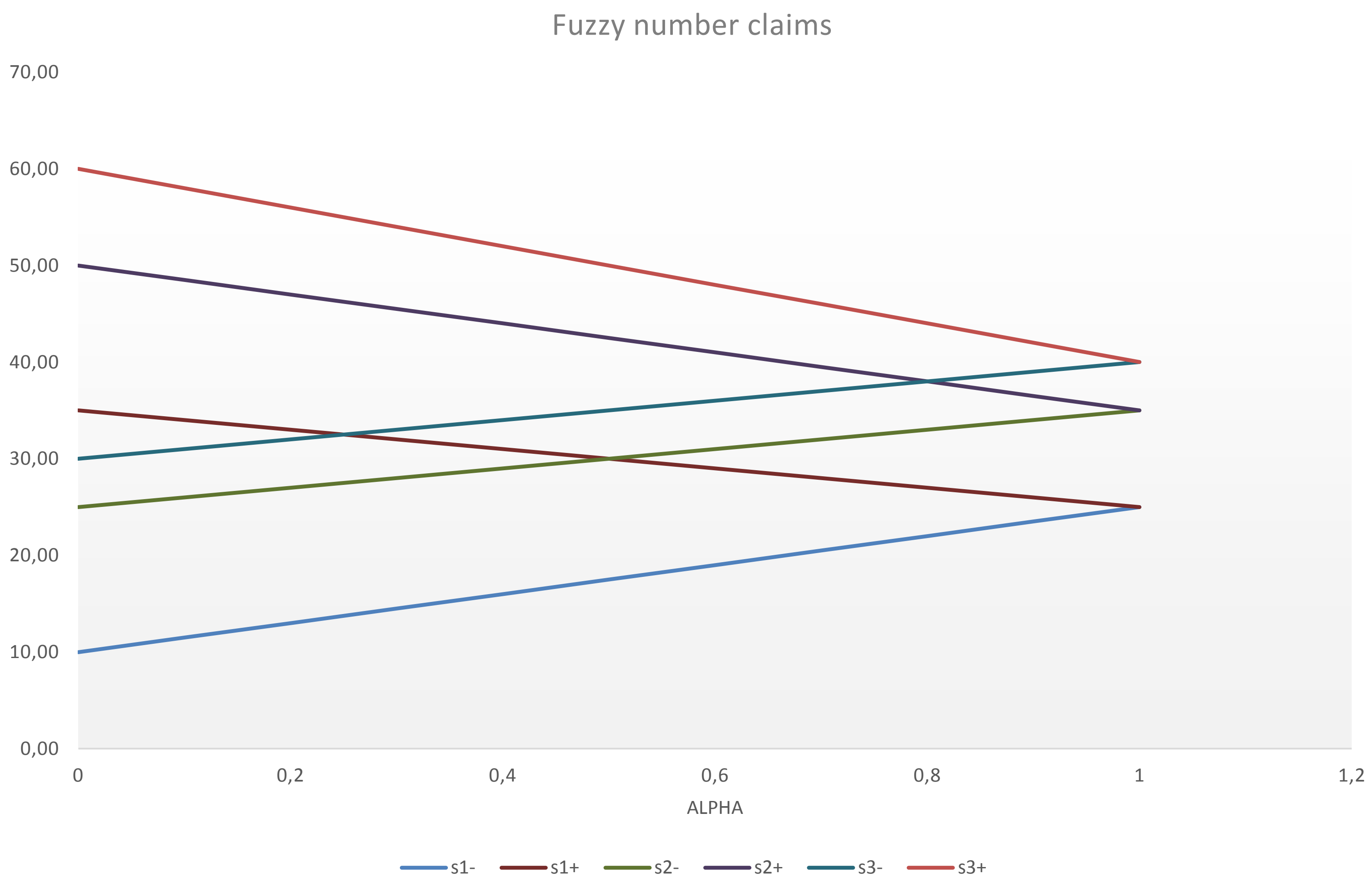
Moreover, $S_i(\widetilde{E})$ is the *crisp* corresponding division share of claimant *i* ∈ *N* , defined by

$$S_i(\widetilde{E}) = \frac{s_i^-(\widetilde{E}) + s_i^+(\widetilde{E})}{2},\ \ i \in N. \qquad (3)$$

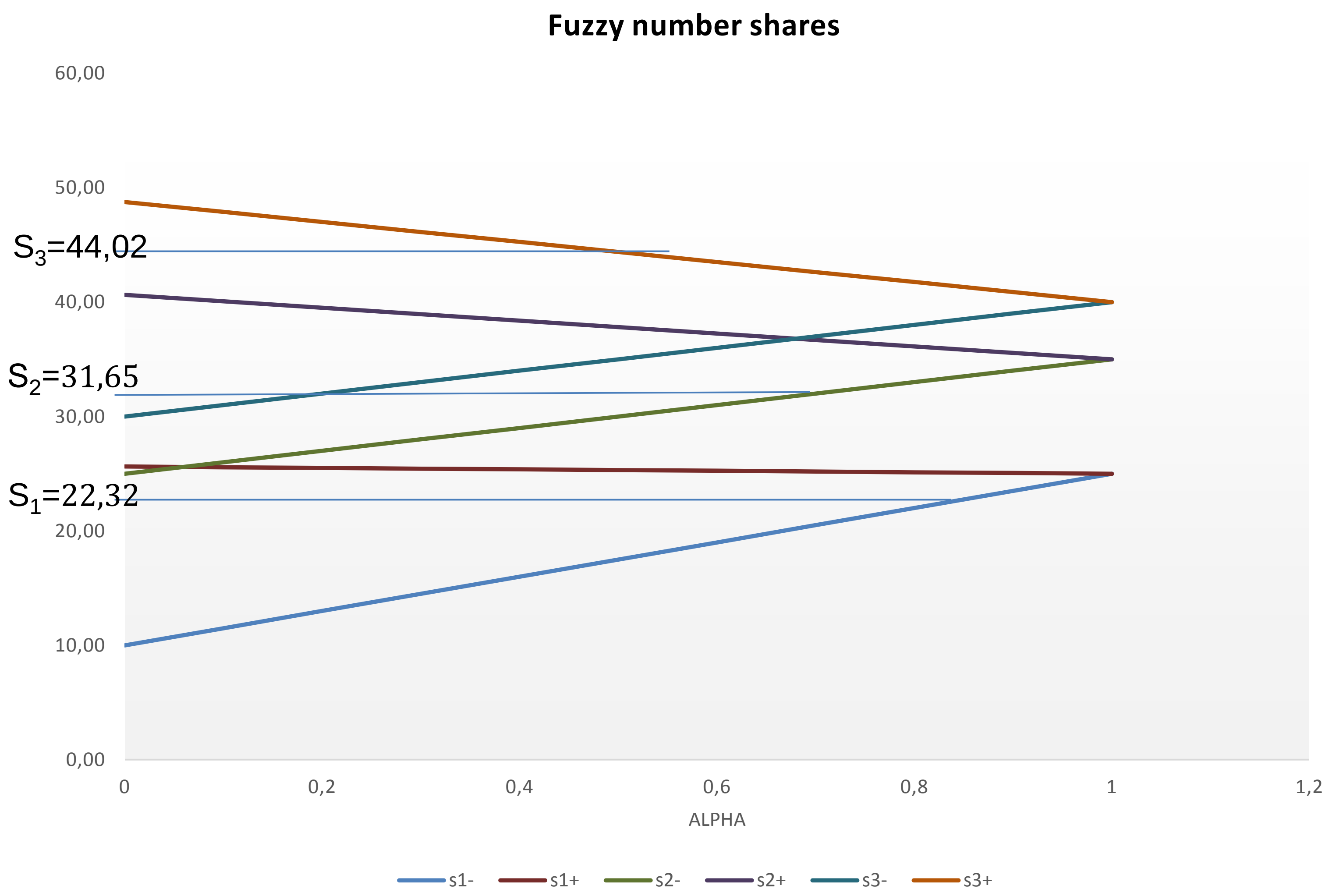
Example: 1

- Let $(\tilde{\boldsymbol{c}}; \widetilde{E})$ be a FB-problem as follows. The following claims are expressed as trapezoidal fuzzy intervals (fuzzy numbers):
- $N = \{1,2,3\}$, $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) \in F(\mathbf{R}^+)^3$, where
- $\tilde{c}_1 = [c_{11}; c_{12}; c_{13}; c_{14}] = [10; 25; 25; 35]$,
- $\tilde{c}_2 = [c_{21}; c_{22}; c_{23}; c_{24}] = [25; 35; 35; 50]$,
- $\tilde{c}_3 = [c_{31}; c_{32}; c_{33}, c_{34}] = [30; 40; 40; 60]$.
- Fuzzy estate is also a trapezoidal fuzzy number
- $\widetilde{E} = [E_1; E_2; E_3; E_4] = [85; 100; 100; 115]$.
- For $\alpha \in [0;1]$ the equivalent formulas by α-cuts are as follows
- $\tilde{c}_1 = [10 + 15\alpha ; 35 - 10\alpha]$, $\tilde{c}_2 = [25 + 10\alpha ; 50 - 15\alpha]$,
- $\tilde{c}_3 = [30 + 10\alpha ; 60 - 20\alpha]$.
- $\widetilde{E} = [85 + 15\alpha ; 115 - 15\alpha]$, see Fig. 1.

Example: 2 (Fig. 1)



Example: 3 (Fig. 2)



Example: 4

- Substituting these values into formulas (2) and (3), we obtain functions $s_i^-(\alpha)$ and $s_i^+(\alpha)$, see Fig. 2. Hence, by these formulas we calculate the integrals of the interval share of each claimant *i* as
- $[s_1^-(\widetilde{E});\ s_1^+(\widetilde{E})] = [19{,}00\ ;\ 25{,}25]$,
- $[s_2^-(\widetilde{E}); s_2^+(\widetilde{E})] = [31{,}00\ ;\ 37{,}25]$,
- $[s_3^-(\widetilde{E}); s_3^+(\widetilde{E})] = [36{,}00\ ;\ 43{,}50]$.
- Moreover, by (3) we obtain the crisp division share of each claimant $i \in N = \{1,2,3\}$, as
- $S_1(\widetilde{E})$ = 22,32 ; $S_2(\widetilde{E})$ = 31,65 ; $S_3(\widetilde{E})$ = 44,02 .
- The above mentioned division scheme $\boldsymbol{s}(,\widetilde{E})$ is an interval solution of the given FB-problem $(\tilde{\boldsymbol{c}}, \widetilde{E})$. Moreover, by the vector of mean values $\boldsymbol{S}(\widetilde{E})$ = (22,32 , 31,65 , 44,02) we obtain a crisp solution of FB-problem $(\tilde{\boldsymbol{c}}, \widetilde{E})$.

Conclusion

- When claims of claimants had fuzzy interval uncertainty, we settled such type of division problems by transforming it into division problems under classical interval uncertainty.
- An example was presented to illustrate particular problems and solution concepts. Here, we extended the classical bankruptcy problem (CB-problem), and the corresponding proportional rule (AP-rule) to FB-problem.
- The other classical bankruptcy rules, e.g. contested garment consistent rule (CGC-rule) and recursive completion rule (RC-rule) could be also extended to FB-problem in the future research.

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